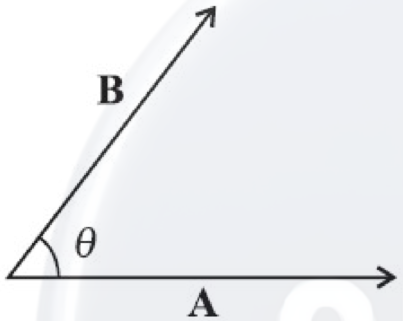
Chapter - 6 Work, Energy And Power

Multiplication of two vectors

1. Scalar product (dot product) – gives a scalar from two vectors
2. Vector product (cross product) – gives a new vector

Scalar product (Dot product) of two vectors



- The scalar product is defined as

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

- Thus

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}$$

Special cases

- If Θ=00, the scalar product is maximum,

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|$$

- If Θ=900, the scalar product is minimum,

$$\vec{A} \bullet \vec{B} = 0$$

Properties of scalar product

- Scalar product is commutative.

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

- Obeys distributive law

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C})$$

- For multiplication with any real number

$$\vec{A} \bullet \left(\lambda \vec{B}\right) = \lambda \left(\vec{A} \bullet \vec{B}\right)$$

- for the orthogonal unit vectors

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$$

- If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

- **Also** $\vec{A} \bullet \vec{A} = |\vec{A}||\vec{A}|\cos 0 = A^2$

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

PROBLEM

- Find the angle between force

$\vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ unit and displacement
$\vec{d} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ unit.

Solution

- We have

$$\cos\theta = \frac{\vec{\mathrm{F}} \bullet \vec{d}}{|\vec{\mathrm{F}}||\vec{\mathrm{d}}|}$$

- **Thus**

$$\vec{\mathrm{F}} \bullet \vec{\mathrm{d}} = F_x d_x + F_y d_y + F_z d_z$$
$$= (3\times5) + (4\times4) + (-5\times3) = 16unit$$
$$|\vec{\mathrm{F}}| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{50}$$
$$|\vec{\mathrm{d}}| = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{50}$$

- Therefore

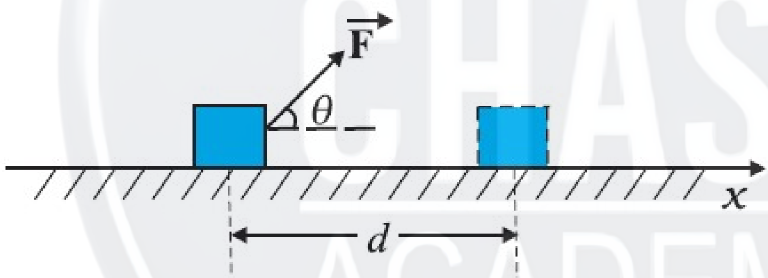
$$\cos\theta = \frac{16}{\sqrt{50}\times\sqrt{50}} = 0.32$$

$$\theta = \cos^{-1} 0.32$$

WORK

- The work done by the force is defined to be the product of component of the force in the direction of the displacement and the magnitude of this displacement.
- The work and energy have the same dimensions [ML-2 T-2]
- The SI unit is joule (J).

Work done by a constant force



- **The work done by the constant force F , is**
- $W = Fd\cos\theta$
- Or $W =$ $\cos\theta = \frac{\vec{F}\bullet\vec{d}}{|\vec{F}||\vec{d}|}$
- Work done can be zero, positive or negative

Special cases

- If ϴ=0, then maximum work is done given by $W = Fd$.

- If Ө=900, then work done =0
- If Ө is between 00 and 900, the work done is positive.
- If Ө is between900 and 1800, the work done is negative.

Situations in which Work done = 0 the displacement is zero (d=0):

• A weightlifter holding a 150 kg mass steadily on his shoulder for 30 s does no work on the load during this time.

the force is zero (F=0):

• A block moving on a smooth horizontal table is not acted upon by a horizontal force (since there is no friction), but may undergo a large displacement

the force and displacement are mutually perpendicular (Ө = 900)

• For the block moving on a smooth horizontal table, the gravitational force *mg does no work since it acts at right* angles to the displacement.

Situations in which work done is negative

- A ball is thrown in the upward direction – work done by the gravitational force is negative.
- The work done by the frictional force, when we push the book to a distance is negative
- The work done by the gravitational force, when we are lifting a bucket of water from the well is negative

PROBLEM

- A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion.

1. How much work does the road do on the cycle?
2. How much work does the cycle do on the road ?

Solution

1. Work done on the cycle by the road = the work done by the frictional force on the cycle

- Here ϴ =1800, F=200N, d=10m ,thus *W* = *Fd* cos ϴ = 200 x 10 x cos1800 = -2000*J*

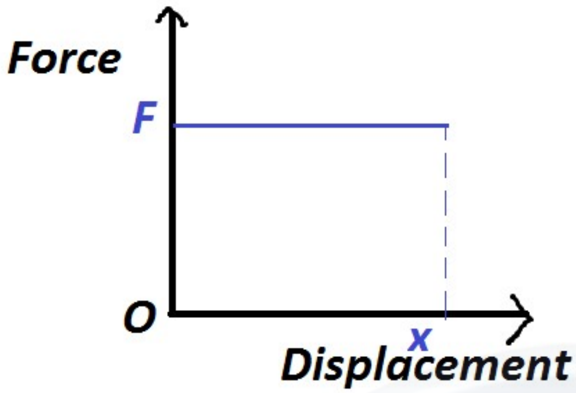
1. The magnitude of the force on the road due to cycle is 200 N.

- The displacement of the road = 0
- Thus, work done by cycle on the road is zero.

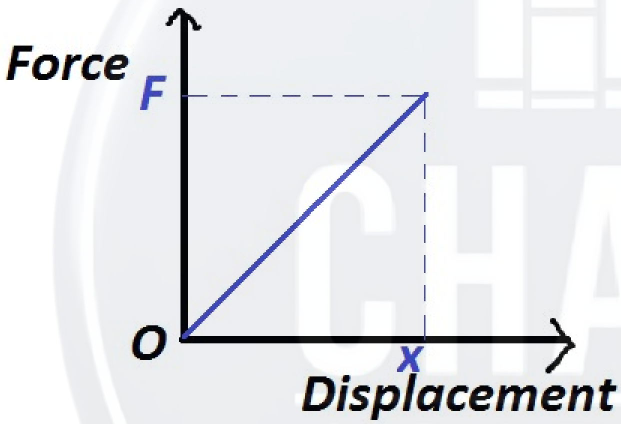
Force –Displacement graph (F-d Graph)

- A graph drawn with displacement along X –axis and force along Y-axis.
- **Area** under F-d graph gives **the work done**.

F-d graph of work done by a constant force

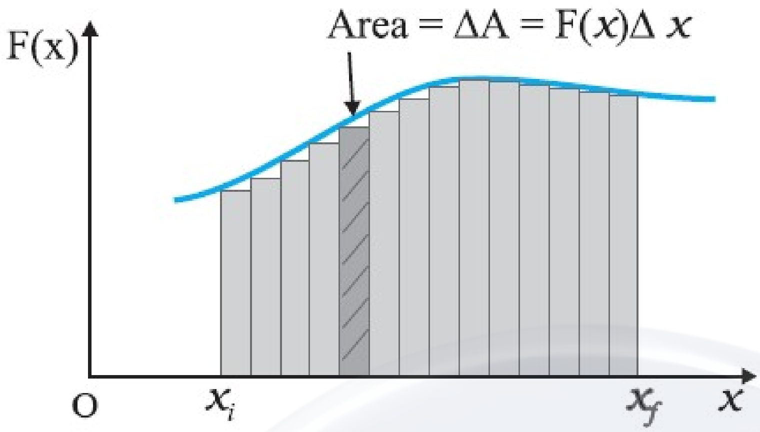


F-d graph of work done by a uniformly varying force



WORK DONE BY A VARIABLE FORCE

- The F-d graph of a varying force in one dimension, is given by



- For a small displacement Δx, the force F(x) is almost costant.
- Thus, the workdone is

$$\Delta W = F(x)\Delta x$$

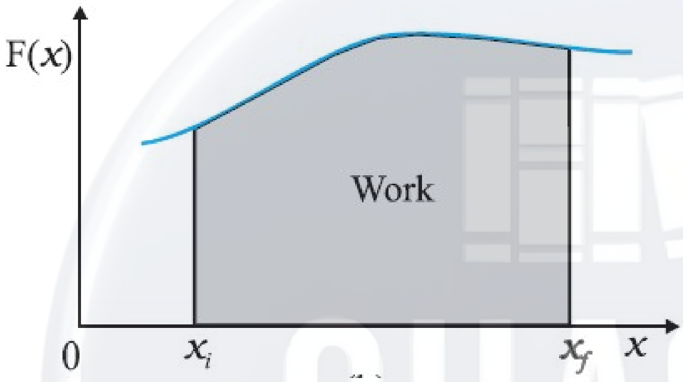
- Therefore, the total work done is given by

$$W = \sum_{x_i}^{x_f} F(x)\Delta x$$

- **Or , if Δ*x* → 0, then**

$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F(x)\Delta x = \int_{x_i}^{x_f} F(x)dx$$

$$W = \int_{x_i}^{x_f} F(x)dx$$



PROBLEM

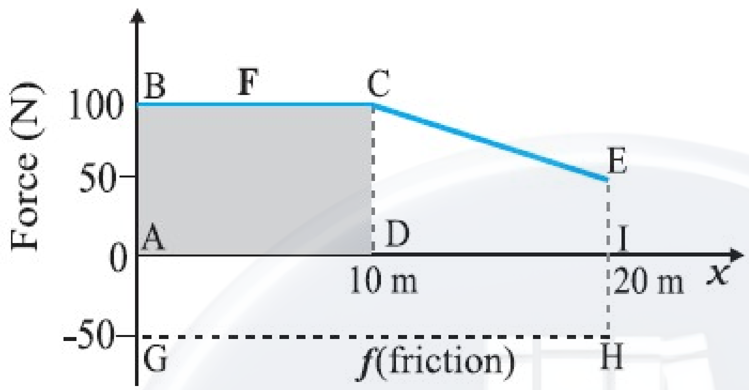
A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100 N over a distance of 10 m. Thereafter, she gets progressively tired and her applied force reduces linearly with distance to 50 N. The total distance through which the trunk has been moved is 20 m.

1. Plot the force applied by the woman and the frictional force which is 50 N.
2. Calculate the work done by the two forces

over 20 m.

Solution

1. The graph is



1. The work done by the women is WF= Area of rectangle ABCD + area of the trapezium CEID

$$W_F = 100 \times 10 + \frac{1}{2}(100 + 50) \times 10 = 1750J$$

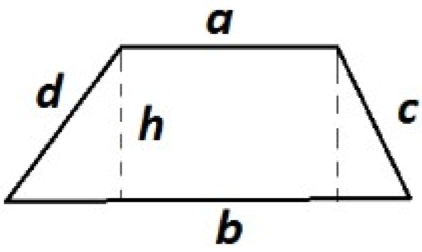
Work done by frictional force is

W_f= Area of rectangle AIHG

$$W_f = -50 \times 20 = -1000J$$

NB: The area of trapezoid is

$A = \frac{1}{2}(a+b) \times h$



ENERGY

- Energy is the capacity for doing work.
- It can be measured by the work that the body can do.
- Joule is the SI unit of energy.

Alternative units of Work /Energy

erg	**10-7J**
electron volt (eV)	**1.6 x 10-19 J**
calorie (cal)	**4.186 J**
kilowatt hour (kWh)	**3.6 x 106 J**

MEHANICAL ENERGY

- The energy of an object due to its motion or position.
- Total mechanical energy is the sum of kinetic and potential energy

KINETIC ENERGY

- The kinetic energy of an object is a measure of the work an object can do by the virtue of its motion
- Kinetic energy an object of mass *m moving with* velocity **v***,* is

$$K = \frac{1}{2}m(\vec{v}.\vec{v}) = \frac{1}{2}mv^2$$

- Kinetic energy is a scalar quantity.
- In terms of momentum , p

$$K = \frac{p^2}{2m}$$

- The dimensions are [ML2T-2]
- The SI unit is joule (J).

WORK- ENERGY THEOREM FOR A CONSTANT FORCE

The change in kinetic energy of a particle is equal to the work done on it by the net force

Proof

- We have , *v2 –u2 =2ad* ,
- Where u – initial velocity , d – displacement, v – final velocity
- Multiplying both sides by *m/2,*

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas = Fs$$

- *In general*

$$v^2 - u^2 = 2\vec{a} \bullet \vec{d}$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = m\vec{a} \bullet \vec{d} = \vec{F} \bullet \vec{d} = W$$

- This is known as **work-energy theorem**

POTENTIAL ENERGY

- Potential energy is the 'stored energy' by virtue of the position or configuration of a body.
- Eg: energy in a stretched string
- The potential energy is released in the form of kinetic energy.
- It is a scalar quantity.
- The dimensions of potential energy are [ML2T-2].
- The SI unit is joule (J).

Gravitational Potential Energy (V)

• Gravitational potential energy of an object at a height h, is the negative of work done by the gravitational force in raising the object to that height.

$$V(h) = mgh$$

- The gravitational force can be written as

$$F = -\frac{dV(h)}{dh} = -mg$$

- Thus, the gravitational force *F* equals the negative of the derivative of *V*(*h*) with respect to *h*.
- The negative sign indicates that the gravitational force is downward.

Show that the gravitational potential energy of the object at height *h*, manifests itself as kinetic energy of the object on reaching the ground.

Proof

- The speed of an object released from a height h, when it just hits the ground is given by

$$v^2 - 0^2 = 2gh = 2gh$$
$$v^2 = 2gh$$

- Multiplying this equation with m/2 on both sides

$$\frac{1}{2}mv^2 = \frac{m}{2} \times 2gh = mgh$$
$$\frac{1}{2}mv^2 = mgh$$

- Thus the kinetic energy = gravitational potential

energy.

Conservative Force

• A force *is conservative if*

1. *it can be derived* from a scalar quantity *V(x).*
2. the work done by the force depends only on initial and final positions.

- Examples are, gravitational force, electric force , spring force etc
- The work done by a conservative force in a closed path is zero.
- The change in potential energy of a conservative force is equal to the negative of the work done by the force.

$$\Delta V = -F(x)\Delta x$$

Show that the work done by a conservative force depends only on end points.

- If the force F(x) is conservative, we can write

$$F(x) = -\frac{dV}{dx}$$

- Thus

$$F(x)dx = -dV$$

$$\int_{x_i}^{x_f} F(x)dx = -\int_{V_i}^{V_f} dV = V_i - V_f$$

- Therefore work done depends only on end points.

Non conservative forces

- The forces in which the work done depends on the factors like velocity or path taken.
- Example: frictional force, viscous force etc.

PRINCIPLE OF CONSERVATION OF MECHANICAL ENERGY

- **The total mechanical energy of a system is conserved if the forces, doing work on it, are conservative.**
- **If forces are conservative**

$$K + V = \text{constant}$$

Proof

- If a body undergoes displacement Δx , under the action of conservative forces, F(x), from work – energy theorem,

$$\Delta K = F(x)\,\Delta x$$

- The change in potential energy is given by

$$\Delta V = -F(x)\Delta x$$

- Adding the two equations

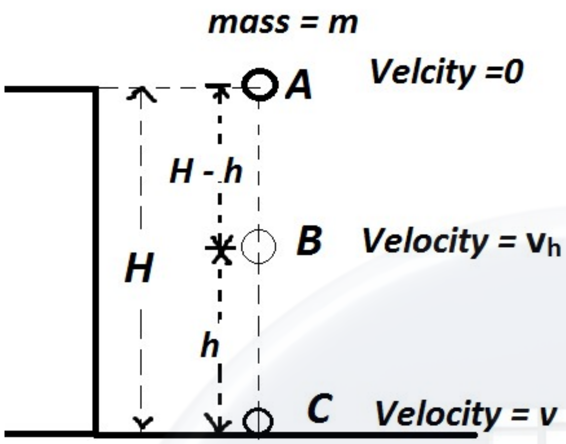
$$\Delta K + \Delta V = F(x)\,\Delta x - F(x)\,\Delta x = 0$$

$$\Delta(K+V) = 0$$

$$K+V = \text{constant}$$

Conservation of Mechanical Energy in a Freely Falling Body

• Consider a ball of mass *m being dropped from a cliff of* height *h.*



Total Energy at the point A

- Kinetic energy at A is zero (K=0), since v=0
- Potential energy at A is , V = mgH
- Thus total energy at A, is

***E*=*K*+*V* =0+mgH =mgH**

Total Energy at the point B

- Kinetic energy at B is

$$K = \frac{1}{2} m v_h^{\ 2}$$

But we have

$$v_h{}^2 - 0^2 = 2g(H-h)$$
$$v_h{}^2 = 2g(H-h)$$

- Thus

$$K = \frac{1}{2}mv_h{}^2 = mg(H-h)$$

- Potential energy at B is , V = mgh
- The total energy at B is

***E*=*K*+*V* =mg(H -h)+mgh =mgH**

Total Energy at the point C

- The kinetic energy at C is

$$K = \frac{1}{2}mv^2$$

- But we have

v*2 -02 = 2*gH

v*2 = 2*gH

- Thus

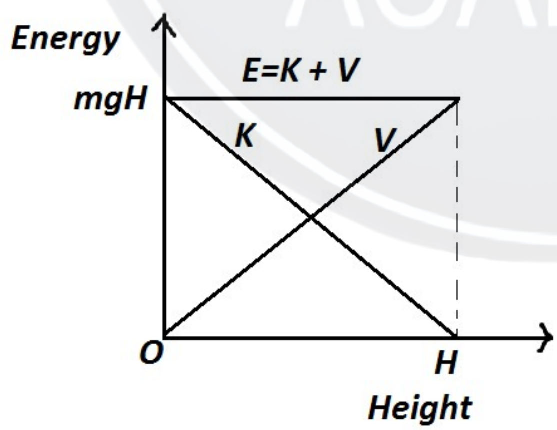
$$K = \frac{1}{2}mv^2 = mgH$$

- The potential energy at C is, V = 0
- The total energy at C

***E*=*K*+*V* =mgH +0=mgH**

- **Therefore total energy at A = total energy at B = Total energy at C = mgH = a constant**

Graph of the variation of kinetic energy and potential energy of a freely falling body



THE POTENTIAL ENERGY OF A SPRING

- The work done by an external force to compress or extend a spring is stored as the potential energy in it.

Spring Force (Fs)

- The restoring force developed in a spring ,when a force is applied on it is called spring force.
- The spring force is a variable force which is conservative.

Force law of spring (Hooke's law)

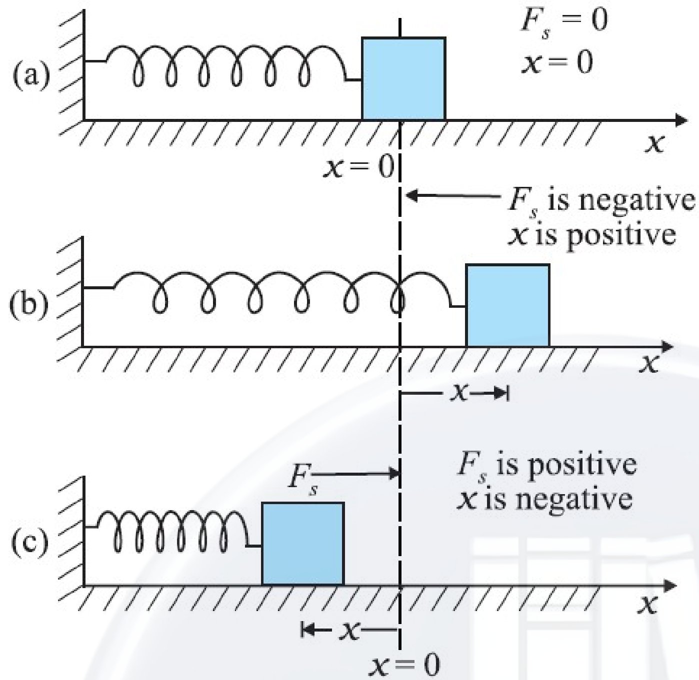
- In an ideal spring, the spring force *F*s is proportional to the displacement.

***Fs* =-kx**

- Where k – spring constant
- Unit of spring constant is N/m.
- The negative sign shows that spring force is opposite to the displacement.

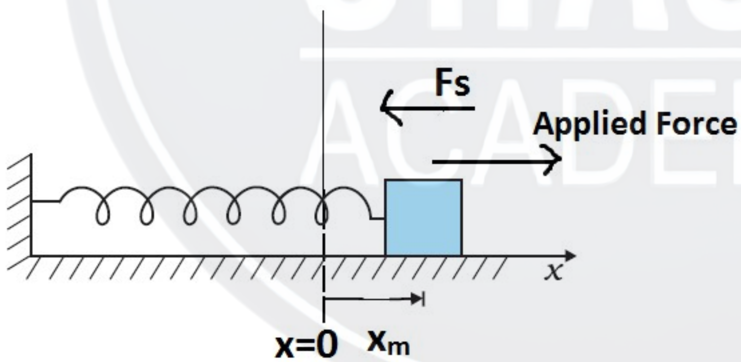
Spring is **stiff** – k is **large**

Spring is **soft** – *k* is **small.**



Equation for the potential energy of a spring

Integration method



The work done by the spring force for an extension of xm is

$$W_s = \int_0^{x_m} F_s dx = -\int_0^{x_m} kx dx$$

$$W_s = -\frac{1}{2}k{x_m}^2$$

- Thus, work done by the applied force will be

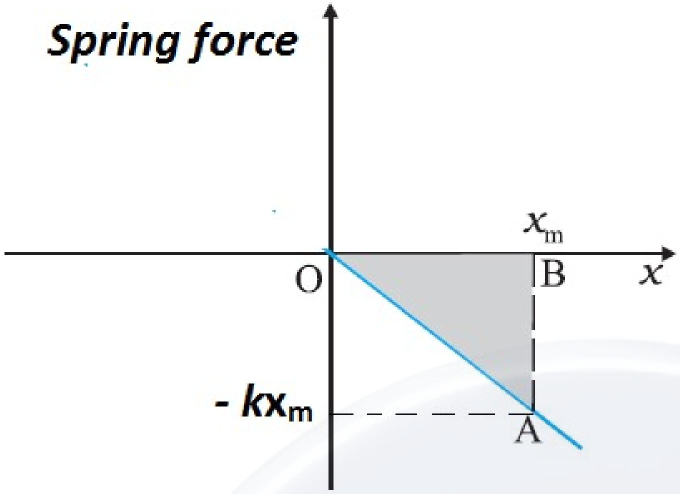
$$W_s = +\frac{1}{2}k{x_m}^2$$

- Therefore, this work done is stored as the potential energy in the spring.
- Potential energy of spring is

$$V = \frac{1}{2}k{x_m}^2$$

ii) Graphical method

- The force – displacement graph of the spring is



- Area under the graph = work done by the spring force
- Thus

$$Area = -\frac{1}{2}kx_m^{\ 2} = W_s$$

- Thus, work done by the applied force is

$$W_s = +\frac{1}{2}kx_m^{\ 2}$$

- Therefore, potential energy of the spring is

$$V = \frac{1}{2}k{x_m}^2$$

Show that spring force is a conservative force

Proof

- **The work done by spring force for a displacement from xi to xf , is *x***

$$W_s = -\int_{x_i}^{x_f} kxdx = \frac{1}{2}k{x_i}^2 - \frac{1}{2}k{x_f}^2$$

- Thus, the work done by spring force depends only on end points.
- Also, the spring force can be derived from a scalar function V(x).

$$V(x) = \frac{kx^2}{2}$$

- That is

$$-\frac{dV(x)}{dx} = -kx$$

- Thus spring force is a conservative force.

Conservation of energy in a spring

- When the block attached to the spring is extended to xm and released, the block oscillates between – xm and + xm
- Thus, the kinetic energy at any point x is given by

$$K = \frac{1}{2}mv^2$$

- The potential energy at x is

$$V = \frac{1}{2}kx^2$$

- Thus the total mechanical energy is

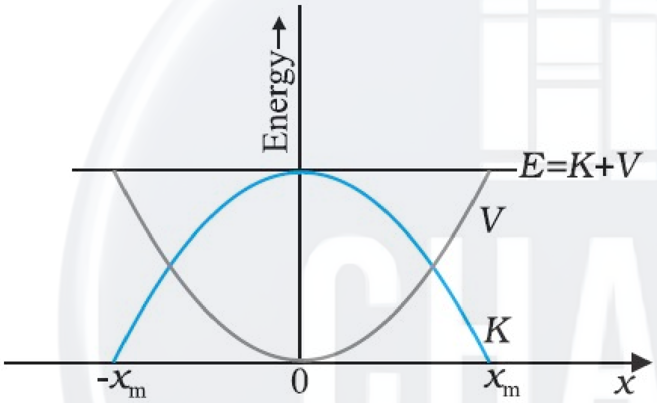
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_m{}^2$$

- Thus the **speed and the kinetic energy** will be **maximum** at the equilibrium position, ***x* = 0**.
- Therefore at x = 0, v = vm

$$\frac{1}{2}m{v_m}^2 = \frac{1}{2}k{x_m}^2$$

$$v_m = \sqrt{\frac{k}{m}}x_m$$

Graph of variation of potential energy and kinetic energy of a spring with displacement



VARIOUS FORMS OF ENERGY

Heat Energy

- Work done by resistive forces like friction etc, is transferred as heat energy.

Chemical Energy

- Chemical energy arises from chemical reactions.
- Chemical reactions may be

1. Exothermic – heat released

2. Endothermic – heat absorbed

Electrical Energy

- Energy associated with electric current.

The Equivalence of Matter and Energy

- Albert Einstein showed that mass and energy are equivalent and are related by the relation

***E*=*mc*2**

- Where *c*, the speed of light in vacuum is approximately 3 x 108m /s.
- A large amount of energy is associated with matter.

Nuclear Energy

- Energy associated with nuclei of an atom.
- Energy can be released by

1. Nuclear fission
2. Nuclear fusion

THE PRINCIPLE OF CONSERVATION OF ENERGY

- Energy can neither be created, nor destroyed but may be transformed from one form to another. The total energy of an isolated system remains constant.
- The total energy of the universe is constant. If one part of the universe loses energy, another part must gain an equal amount of

energy.

POWER

- **Power** is defined as the time rate at which work is done or energy is transferred.
- Average power is given by

$$P_{av} = \frac{W}{t}$$

- Where W – total work done, t – total time
- **The instantaneous power** is given by

$$P_{av} = \frac{W}{t}$$

- Power is a **scalar** quantity
- SI unit – **watt (W)**
- Dimensions are **[ML2T-3]**
- Another unit of power is **horse – power (hp)**

1*hp* = 746*W*

- Horsepower is used to describe the output of automobiles, motorbikes, etc.

Relation connecting power, force and velocity

- We have the work done

$$dW = \vec{F} \bullet d\vec{r}$$

- Where F – force , dr – displacement.
- Thus the instantaneous power is given by

$$P = \frac{dW}{dt} = \vec{F} \bullet \frac{d\vec{r}}{dt}$$

- That is

$$P = \vec{F} \bullet \vec{v}$$

Unit of electrical energy

Electrical energy is often expressed in kilowatt hour (kWh)

1*kWh* = 3.6×106*J*

PROBLEM

An elevator can carry a maximum load of 1800 kg (elevator + passengers) is moving up with a constant speed of 2 m s–1. The frictional force opposing the motion is 4000 N. Determine the minimum power delivered by the motor to the elevator in watts as well as in horse power.

Solution

- Downward force on the elevator is

$$F = mg + F_f = (1800 \times 10) + 4000 = 22000\ \text{N}$$

- The power is

$$P = \mathbf{F}.\ \mathbf{v} = 22000 \times 2 = 44000\ \text{W} = 59\ \text{hp}$$

COLLISIONS

- A collision is an isolated event in which, two or more bodies exert strong forces on each other for a short time.

Types of collisions

Elastic collisions

- The momentum and kinetic energy are conserved.
- It is an ideal collision.

Completely inelastic collisions

- Collision in which the two particles move together after the collision.
- The momentum is conserved, but kinetic energy is not conserved

Inelastic collision

- The momentum is conserved, but kinetic energy is not conserved.
- Most of the collisions are inelastic.

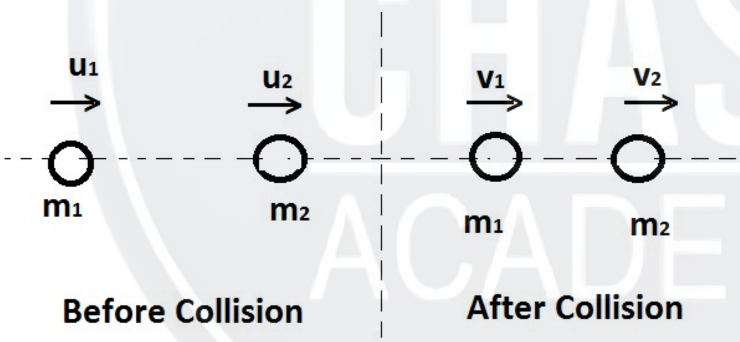
One dimensional collision (head- on collision)

- Velocity of particles before collision and after collision is directed along the same straight line.

Two dimensional collision

- Velocities of particles lie in a plane.

ELASTIC COLLISION IN ONE DIMENSION



- If $u_1 > u_2$, the bodies collides each other.
- According to principle of conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \rightarrow (1)$$

- According to principle of conservation of energy

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \rightarrow (2)$$

- From equation (1)

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \rightarrow (3)$$

- From equation (2)

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \rightarrow (4)$$

- Dividing equations (4) and (3)

$$\frac{(u_1^2 - v_1^2)}{(u_1 - v_1)} = \frac{(v_2^2 - u_2^2)}{(v_2 - u_2)}$$

On simplification

$$\frac{(u_1+v_1)(u_1-v_1)}{(u_1-v_1)}=\frac{(v_2+u_2)(v_2-u_2)}{(v_2-u_2)}$$

$$(u_1+v_1)=(v_2+u_2)\rightarrow(5)$$

$$(v_1-v_2)=(u_2-u_1)=-(u_1-u_2)\rightarrow(6)$$

Thus the relative velocity after collision is numerically equal to relative velocity before collision.

- From equation (5)

$$v_2=u_1-u_2+v_1$$

- Substituting this in equation (1)

$$m_1u_1+m_2u_2=m_1v_1+m_2(u_1-u_2+v_1)$$
$$=m_1u_1-m_2u_1+m_2u_2+m_2u_2=m_1v_1+m_2v_1$$
$$(m_1-m_2)u_1+2m_2u_2=(m_1+m_2)v_1$$

- Therefore

$$v_1=\left(\frac{m_1-m_2}{m_1+m_2}\right)u_1+\left(\frac{2m_2}{m_1+m_2}\right)u_2$$

- Similarly

$$v_2=\left(\frac{m_2-m_1}{m_1+m_2}\right)u_2+\left(\frac{2m_1}{m_1+m_2}\right)u_1$$

Special cases

i) If m2 is initially at rest (u2 = 0), then

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1$$
$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1$$

ii) If m1 = m2 = m

$$v_1 = 0 \times u_1 + \left(\frac{2m}{2m}\right)u_2 = u_2$$
$$v_2 = \left(\frac{2m}{2m}\right)u_1 = u_1$$

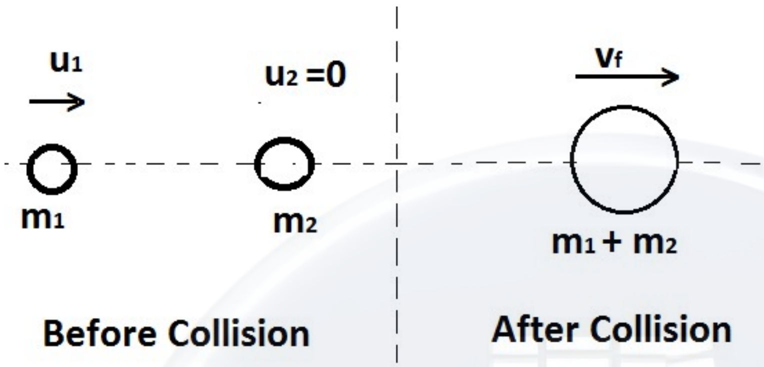
Thus, the **velocities are exchanged.**

iii) If m1 = m2 = m and u2 = 0

$$v_1 = 0 \times u_1 + \left(\frac{2m}{2m}\right) \times 0 = 0$$
$$v_2 = 0 \times u_2 + \left(\frac{2m}{2m}\right)u_1 = u_1$$

- Thus the **first body comes to rest** and **second body attains the** velocity of the first body

COMPLETELY INELASTIC COLLISION IN ONE DIMENSION



- According to principle of conservation of momentum

$$m_1 u_1 + 0 = (m_1 + m_2) v_f$$

- Thus

$$v_f = \left(\frac{m_1}{m_1 + m_2} \right) u_1$$

- The loss of kinetic energy in collision is

$$\Delta K = \frac{1}{2} m_1 u_1^2 - \frac{1}{2} (m_1 + m_2) v_f^2$$

- Substituting for vf ,

$$\Delta K = \frac{1}{2} m_1 u_1^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}\right)^2 u_1^2$$

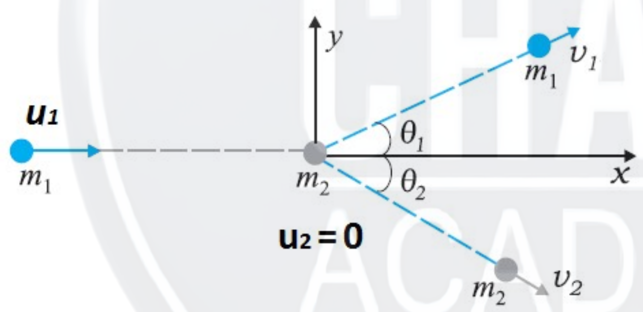
$$= \frac{1}{2} m_1 u_1^2 - \frac{1}{2}\left(\frac{m_1^2}{m_1 + m_2}\right) u_1^2$$

$$\Delta K = \frac{1}{2} m_1 u_1^2 \left(1 - \left(\frac{m_1}{m_1 + m_2}\right)\right)$$

$$\Delta K = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) u_1^2$$

Thus change in kinetic energy is a positive quantity.

COLLISIONS IN TWO DIMENSIONS



- In a two dimensional collision momentum along x and y direction should be conserved.

- Along x – direction ,

$$m_1u_1 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2$$

- Along y – direction

$$0 = m_1v_1\sin\theta_1 - m_2v_2\sin\theta_2$$

- If the collision is elastic , the conservation f kinetic energy gives

$$\frac{1}{2}m_1{u_1}^2 = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$$